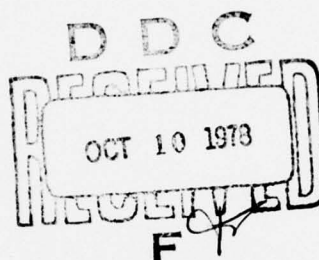# LEVEL II

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Technical Report 232

# LINEAR PREDICTIVE SPECTRAL ESTIMATION OF BANDLIMITED SIGNALS IN LOWPASS NOISE

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| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>NOSC/TR-232 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE *(and Subtitle)*<br>LINEAR PREDICTIVE SPECTRAL ESTIMATION OF BANDLIMITED SIGNALS IN LOWPASS NOISE | | 5. TYPE OF REPORT & PERIOD COVERED<br>Research<br>January to April 1977 |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>S.T. Alexander, E.H. Satorius | | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Naval Ocean Systems Center<br>San Diego, CA 92152 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>62611N, F11101,<br>XF111011100 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Naval Electronic Systems Command<br>Washington, D.C. | | 12. REPORT DATE<br>31 March 1978 |
| | | 13. NUMBER OF PAGES<br>20 |
| 14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)* | | 15. SECURITY CLASS. *(of this report)*<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Linear Prediction Filtering
Method of Undetermined Coefficients
Bandlimited Signals
Lowpass Correlated Noise

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The discrete Wiener matrix equation is solved for the case of a complex bandlimited (BL) signal corrupted by additive lowpass correlated noise. The solution is achieved by the method of undetermined coefficients which significantly reduces the dimensionality of the problem. Asymptotic properties of the minimum mean-square error (MMSE) weight vector solutions are examined for the cases of very long filter lengths and degeneration to white noise. These special cases allow the MMSE filter impulse response to be examined in terms of its noise and signal components. It is then shown that by proper choice of prediction distance $\Delta$, one may remove much of the noise effects upon the MMSE filter while retaining the desired BL signal characteristics.

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# CONTENTS

## INTRODUCTION

The use of minimum mean-square error (MMSE) filtering for prediction, estimation and smoothing has grown in popularity during the last few years. To a large degree this interest may be attributed to the corresponding growth in the areas of real-time computing hardware and software. The degree of sophistication reached in these areas has made it feasible to construct dedicated hardware componentry or interactive software algorithms capable of implementing MMSE filtering.

As originally developed by Wiener,[1] the theoretical techniques of MMSE filtering permitted us to solve analytically for the linear filter impulse response. This would minimize the mean-squared error between the filter's output signal and a desired signal. We have considerable freedom in choosing this desired signal; the specific application determines its choice.

One possibility enables us to use the output of a linear filter and to predict the future values of a time series when we know its present and past values. The optimal filter impulse response (which minimizes the mean square of the error between the filter output and the selected time series value) appears (in the continuous case) as the solution to a Fredholm integral equation of the first kind. Solution strategies for this equation have been developed by a number of authors[2,3,4] for several signal and noise situations. For the case of discrete data, the solution for the optimal L-length filter coefficients appears as the solution vector, $\underline{w}^*$, to the well-known discrete Wiener-Hopf matrix equation:

$$R \underline{h} = \underline{P}_\Delta. \tag{1}$$

In Equation 1, $R$ is the $L \times L$ autocorrelation matrix of the input time series, $\underline{h}$ is the $L \times 1$ vector of the optimal weight coefficients (the discrete impulse response of the filter) and $\underline{P}_\Delta$ is the $L \times 1$ autocorrelation vector with autocorrelation elements

$$\underline{P}_\Delta = \begin{bmatrix} \phi_{XX}(\Delta) \\ \phi_{XX}(\Delta + 1) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{XX}(\Delta + L - 1) \end{bmatrix}. \tag{2}$$

where $\Delta$ is the prediction distance in sampling intervals.

The main difficulty in solving Equation 1 is the estimation of the true autocorrelation lags, $\phi_{XX}(\ell)$, for the time series under examination. This in itself is an area of active current research. In actual systems, we are given a finite amount of data from which to estimate the actual lag values. This introduces some degree of uncertainty. This work examines the theoretical case of assuming the autocorrelation lags known exactly and develops an analytical solution to Equation 1. Solutions developed for this ideal case are useful for comparison with actual processors which approximate solutions to $\underline{h}$ from finite segments of noisy data. Although this is a natural application of the analysis in this paper, we will confine the present work to examining the properties of the theoretical solutions to Equation 1.

The specific case to be examined is that of bandlimited (BL) signals corrupted by lowpass noise. An example is the signal generation by a Gaussian random process of a bandlimited signal and the noise characteristics of the media exhibit the familiar "1F" frequency

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absorption. An analytical technique known as the method of undetermined coefficients is applied toward the solution of Equation 1 for this case. Early theoretical work done by Zadeh and Ragazzini[2] devised this method for continuous systems. The method was adapted to the discrete case by Krut'ko[5] and Solodovnikov.[6] More recently, general properties of the discrete method were examined by Satorius and Zeidler.[7] The discrete method was then applied by Zeidler et al[8] to solving Equation 1 for the case of multiple sinusoids in uncorrelated noise and by Satorius and Zeidler[9] to the case of multiple sinusoids in lowpass noise. To fully examine the effects of a nonzero bandwidth signal upon the resulting optimal filter structure, the input is limited to one complex bandlimited signal in lowpass noise. Motivation for examining this case stems from Reference 8. That work reveals that as long as the frequencies of the complex exponentials comprising sinusoids are sufficiently separated, there is effectively no interference between each complex exponential solution. Solutions corresponding to real sinusoids then become linear superpositions of the complex exponential solutions.

## ANALYSIS

The matrix Equation 1 may be expanded into its L components to become

$$\sum_{k=0}^{L-1} \phi_{xx}(k)\, h(k) = \phi_{xx}(\ell + \Delta); \qquad \ell = 0, 1, \ldots, L - 1. \tag{3}$$

For the specific case of one BL signal in lowpass noise, the form of the autocorrelation lag $\phi_{xx}(\ell)$ becomes

$$\phi_{xx}(\ell) = \sigma_N^2\, e^{-\alpha_N|\ell|} + \sigma_s^2\, e^{-\alpha_s|\ell|}\, e^{j\omega_s\ell} \tag{4}$$

In Equation 4,

$\sigma_N^2$ = noise mean-square power

$\sigma_s^2$ = signal mean-square power

$\alpha_N, \alpha_s$ = correlation parameters of noise and signal, respectively

$\omega_s$ = radian center frequency (relative to Nyquist) of complex BL signal

Following the method outlined in Reference 7, we assume the general solution to Equation 3 for the case of a complex BL signal in lowpass noise to be:

$$h(k) = \sum_{m=1}^{2} B_m\, z_m^k + C_1\, \delta(k) + C_2\, \delta(k - L - 1), \qquad k = 0, 1, \ldots, L - 1. \tag{5}$$

In Equation 5, the $B_m$, $C_1$ and $C_2$ are complex coefficients, as yet undetermined; the $z_m$ are damped complex exponentials of the form $z_m = \exp(\mu_m + j\theta_m)$; and the delta functions $\delta(\cdot)$ are due to the end effects of the finite length filter.

The solution strategy is to substitute Equation 5 for the optimal weight vector and Equation 4 for the autocorrelation function into Equation 3 and solve the set of resulting equations for the unknowns $B_m$, $z_m$, $C_1$ and $C_2$. Applying this technique yields six

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equations in six unknowns, which may then be partitioned according to the following consistency requirements:

1. The coefficients of $z_m^\ell$ in the resultant substitution must be equal for every $\ell$ of the equation;
2. The coefficients of $e^{-\alpha N \ell}$ and $e^{\alpha N \ell}$ must similarly equate and
3. The coefficients of $e^{(-\alpha_s + j\omega_s)\ell}$ and $e^{(\alpha_s + j\omega_s)\ell}$ must also equate

Condition 1 leads to a complex quadratic equation which may be solved for the complex exponentials $z_1$ and $z_2$. Conditions 2 and 3 give a set of four equations in the remaining four unknowns ($B_1$, $B_2$, $C_1$, $C_2$). Thus the unknown parameters of Equation 5 may be found and the resulting optimal weight vector determined.

Applying condition 1 gives the following equation:

$$\sigma_N^2 \left[ \frac{1}{1 - z_m \, e^{-\alpha N}} - \frac{1}{1 - z_m \, e^{\alpha N}} \right]$$
$$+ \sigma_s^2 \left[ \frac{1}{1 - z_m \, e^{-\alpha_s - j\omega_s}} - \frac{1}{1 - z_m \, e^{\alpha_s - j\omega_s}} \right] = 0 \tag{6}$$

which after a few operations leads to

$$\sigma_N^2 \sinh \alpha_N (z_m^2 - 2 \cosh \alpha_s \, e^{j\omega_s} z_m + e^{j2\omega_s})$$
$$+ \sigma_s^2 \, e^{j\omega_s} \sinh \alpha_s (z_m^2 - 2 \cosh \alpha_N z_m + 1) = 0. \tag{7}$$

Equation A17 in the Appendix shows that this is equivalent to the condition

$$N(z_m) = 0 \tag{8}$$

where $N(z)$ is the z-domain polynomial representing the numerator of the power spectral density $S_{xx}(z)$ given by

$$S_{xx}(z) = \frac{N(z)}{D(z)} = Z \{\phi_{xx}(\ell)\} . \tag{9}$$

In Equation 9, $D(z)$ represents the denominator z-polynomial, $\phi_{xx}(\ell)$ is the autocorrelation function from Equation 4 and $Z\{\cdot\}$ signifies the z-transform operation. Equation 8 states that the $z_m$ are located at the zeroes of the input spectral density. For the case of one BL complex line the zeroes are given by the pair $\{z_1, z_2\}$ which are reciprocal to the unit circle on the frequency radial $\theta_1$:

$$z_1 = e^{-\mu} \, e^{j\theta_1} \tag{10a}$$
$$z_2 = e^{\mu} \, e^{j\theta_1}. \tag{10b}$$

In Equation 10, $\mu$ is a measure of the closeness of the zero to the unit circle and for the present case, $\mu > \alpha_s$.

Applying the consistency conditions 2 and 3, we obtain the following set of equations:

$$B_1 \left[ \frac{1}{1 - z_1 \, e^{\alpha N}} \right] + B_2 \left[ \frac{1}{1 - z_2 \, e^{\alpha N}} \right] + C_1 = e^{-\alpha N \Delta} \tag{11a}$$

$$B_1 \left[ \frac{-z_1^L e^{-\alpha NL}}{1 - z_1 e^{-\alpha N}} \right] + B_2 \left[ \frac{-z_2^L e^{-\alpha NL}}{1 - z_2 e^{-\alpha N}} \right] + C_2 e^{-\alpha N(L-1)} = 0 \tag{11b}$$

$$B_1 \left[ \frac{1}{1 - z_1 e^{\alpha_S - j\omega_S}} \right] + B_2 \left[ \frac{1}{1 - z_2 e^{\alpha_S - j\omega_S}} \right] + C_1 = e^{(-\alpha_S + j\omega_S)\Delta} \tag{11c}$$

$$B_1 \left[ \frac{-z_1^L e^{(-\alpha_S - j\omega_S)L}}{1 - z_1 e^{-\alpha_S - j\omega_S}} \right] + B_2 \left[ \frac{-z_2^L e^{(-\alpha_S - j\omega_S)L}}{1 - z_2 e^{-\alpha_S - j\omega_S}} \right] + C_2 e^{-(\alpha_S + j\omega_S)(L-1)} \tag{11d}$$

$$= 0.$$

The substitutions from Equation 10 are then used in Equations 11 from which the unknown coefficients may be determined and the solution for the weight vector becomes

$$h(k) = B_1 e^{-\mu k} e^{j\theta_1 k} + B_2 e^{\mu k} e^{j\theta_1 k} + C_1 \delta(k) + C_2 \delta(k - L + 1),$$

$$k = 0, 1, \ldots, L - 1. \tag{12}$$

## ASYMPTOTIC PROPERTIES

Equation 12, with $\{B_1, B_2, C_1, C_2\}$ determined from Equations 11, gives the general solution for the weight vector for arbitrary filter length L, signal bandwidth parameter $\alpha_S$ and noise bandwidth parameter $\alpha_N$. However, more insight into the properties of the optimal impulse response may be gleaned by considering a few asymptotic properties of the general solution.

## LARGE FILTER LENGTH, L

First, consider the form of the impulse response as the filter length L becomes larger and larger ($L \to \infty$ in the limit). Note that the term $e^{\mu k}$ present in Equation 12 is unbounded for increasing k. For L unbounded, this will cause an unbounded solution for the weight vector. Thus, if we consider the case of long filter length L ($L \to \infty$ in the limit), the coefficients $B_2$ must be equal identically to zero to give a stable solution for h(k). With this simplifying assumption, the Equations 11a and 11c become

$$B_1 \left[ \frac{1}{1 - z_1 e^{\alpha N}} \right] + C_1 = e^{-\alpha N \Delta} \tag{13a}$$

$$B_1 \left[ \frac{1}{1 - z_1 e^{\alpha_S - j\omega_S}} \right] + C_i = e^{(-\alpha_S + j\omega_S)\Delta} \tag{13b}$$

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which may be solved for $B_1$ and $C_1$ to give:

$$B_1 = \left\{ \frac{[1 - e^{\alpha N - \mu} e^{j\theta_1}] \, [1 - e^{\alpha_s - \mu} e^{j(\theta_1 - \omega_s)}]}{e^{\alpha N - \mu} e^{j\theta_1} - e^{\alpha_s - \mu} e^{j(\theta_1 - \omega_s)}} \right\} [e^{-\alpha N \Delta} - e^{(-\alpha_s + j\omega_s)\Delta}] \quad (14a)$$

$$C_1 = e^{-\alpha N \Delta} - \left\{ \frac{[1 - e^{\alpha_s - \mu} e^{j(\theta_1 - \omega_s)}] \, [e^{-\alpha N \Delta} - e^{(-\alpha_s + j\omega_s)\Delta}]}{e^{\alpha N - \mu} e^{j\theta_1} - e^{\alpha_s - \mu} e^{j(\theta_1 - \omega_s)}} \right\} . \quad (14b)$$

Making the approximation $B_2 = 0$ in Equation 11b gives the following expression for $C_2$:

$$C_2 = e^{-\alpha N} \left[ \frac{e^{-\mu L} e^{j\theta_1 L}}{1 - e^{-\alpha N - \mu}} \right] . \quad (15)$$

But for the long filter length case ($L \gg 4/\mu$), the factor $e^{-\mu L}$ asymptotically approaches zero and thus $C_2$ vanishes.

Therefore, for the case of very long filter length the optimal weight vector reduces to the following approximation:

$$h(k) \cong B_1 e^{-\mu k} e^{j\theta_1 k} + C_1 \delta(k) \qquad k = 0, \ldots, L-1 : L \to \infty. \quad (16)$$

The weight vector given by Equation 16 has two parts: a damped complex exponential oscillating at the frequency of the spectral zero of the input process, and an impulse of strength $C_1$, existing in the first weight. The damped exponential decays away from the beginning of the filter with an envelope determined by the value of $\mu$, the zero-damping parameter. For very small $\mu$, the oscillatory function is almost a pure sinusoidal term; for larger $\mu$ the solution decays to zero quickly. The impulse $C_1 \delta(k)$ gives a value $C_1$ to the first ($k = 0$) weight only. Physically, the BL signal may be thought of as causing the damped exponential portion of $h(k)$, while the lowpass noise contributes the impulse with magnitude $C_1 \delta(k)$.

Once the impulse response $h(k)$ has been derived, the transfer function $H(j\omega)$ of the optimal filter may be determined by taking the z-transform of $h(k)$ and evaluating around the unit circle at $z = e^{j\omega}$. Thus from Equation 16,

$$H(z) = Z\{h(k)\} = \sum_{k=0}^{L-1} h(k) z^{-k}$$

$$= B_1 \sum_{k=0}^{L-1} [e^{-\mu + j\theta_1} z^{-1}]^k + C_1$$

or

$$H(z) = B_1 \left[ \frac{1 - e^{-\mu L + j\theta_1 L} z^{-L}}{1 - e^{-\mu + j\theta_1} z^{-1}} \right] + C_1 . \qquad (17)$$

Since $L \to \infty$, the exponential $e^{-\mu L}$ approaches zero. Using this approximation and evaluating at $z = e^{j\omega}$, the transfer function $H(j\omega)$ becomes

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$$H(j\omega) = \frac{B_1}{1 - e^{-\mu} e^{j(\theta_1 - \omega)}} + C_1. \tag{18}$$

Thus, the optimal transfer function has a constant background component of complex value $C_1$ (given by Equation 14b) plus a component due to the BL signal presence. The spectral peak of this latter term is, in general, a function of all the parameters (as seen from Equation 14a) and is not examined further in this paper. However, an enlightening simplication results from considering the BL signal in white noise, which we do next.

## WHITE NOISE APPROXIMATION FOR LONG FILTER LENGTH

White noise of mean-square power $\sigma_N^2$ can be approximated from Equation 7 by allowing $\alpha_N \to \infty$. The $z_m$ for this case may be obtained easily from Equation 7 by first dividing by $\cosh \alpha_N$ and taking the limit as $\alpha_N \to \infty$:

$$\sigma_N^2 z_m^2 - 2 [\sigma_N^2 \cosh \alpha_s + \sigma_S^2 \sinh \alpha_s] e^{j\omega_s} z_m + \sigma_N^2 e^{j2\omega_s} = 0, \tag{19}$$

$$m = 1, 2.$$

From the quadratic formula for complex numbers and letting

$$a = \cosh \alpha_s + \left( \frac{\sigma_S^2}{\sigma_N^2} \right) \sinh \alpha_s \tag{20}$$

Equation 19 has the following solutions:

$$z_1 = e^{j\omega_s} [a - \sqrt{a^2 - 1}] \tag{21a}$$

$$z_2 = e^{j\omega_s} [a + \sqrt{a^2 - 1}]. \tag{21b}$$

But from Equation 10, it has been shown that $z_1$ and $z_2$ appear at the spectral zero locations of the input process. Thus, equating Equation 10a with Equation 21a and 10b with 21b gives the following relation:

$$e^{\pm\mu} e^{j\theta_1} = [a \pm \sqrt{a^2 - 1}] e^{j\omega_s}. \tag{22}$$

Equating the real and imaginary parts of Equation 22:

$$\mu = \ell n [a - \sqrt{a^2 - 1}] \tag{23a}$$

$$\theta_1 = \omega_s. \tag{23b}$$

Furthermore, from Equation 14b we see that as $\alpha_N \to \infty$, $C_1$ approaches zero. Substituting this result and Equations 23 into the asymptotic solution (Equation 16) for h(k), we obtain:

$$h(k) \cong B_1 [a - \sqrt{a^2 - 1}]^k e^{j\omega_s k} \tag{24}$$

$$k = 0, \ldots, L - 1; L \to \infty.$$

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An important property of the impulse response for the BL signal in white noise is seen from Equation 24. Namely, the oscillatory portion of $h(k)$ is at exactly the center frequency, $\omega_S$, of the signal regardless of bandwidth. Moreover, the impulse-type noise contribution due to $C_1$ has disappeared for the limiting case of white noise. Further characteristics of the solution are seen by considering the effects of $\alpha_N \to \infty$ upon $B_1$. This may be examined by first multiplying the numerator and denominator of Equation 14a by $e^{-\alpha_N}$, giving:

$$B_1 = \left\{ \frac{[e^{-\alpha_N} - e^{-\mu} e^{j\theta_1}][1 - e^{\alpha_S - \mu} e^{j(\theta_1 - \omega_S)}]}{e^{-\mu} e^{j\theta_1} - e^{-\alpha_N + \alpha_S - \mu} e^{j(\theta_1 - \omega_S)}} \right\} [e^{-\alpha_N \Delta} - e^{(-\alpha_S + j\omega_S)\Delta}].$$
(25)

Taking the limit as $\alpha_N \to \infty$,

$$\lim_{\alpha_N \to \infty} B_1 = \frac{e^{-\mu} e^{j\theta_1} [1 - e^{\alpha_S - \mu} e^{j(\theta_1 - \omega_S)}] e^{(-\alpha_S + j\omega_S)\Delta}}{e^{-\mu} e^{j\theta_1}}.$$
(26)

But $\theta_1 = \omega_S$ for the white noise case and thus

$$B_1 = e^{-\alpha_S \Delta} (1 - e^{\alpha_S - \mu}) e^{j\omega_S \Delta}.$$
(27)

$B_1$ is thus complex with a magnitude determined by the bandwidth of the signal (due to $\alpha_S$) and delay, $\Delta$, and with an initial phase determined by the signal center frequency $\omega_S$ and delay. When Equation 24 is substituted into Equation 27, the solution for the weight vector becomes

$$h(k) = e^{-\alpha_S \Delta} (1 - e^{\alpha_S - \mu}) e^{-\mu k} e^{j\omega_S(k + \Delta)},$$
(28)

$$0 \leqslant k \leqslant L - 1.$$

The solution is thus seen to be a damped complex exponential with initial phase $\omega_S \Delta$ and initial magnitude $e^{-\alpha_S \Delta} (1 - e^{\alpha_S - \mu})$. For increasing $\Delta$, the magnitude decreases as $e^{-\alpha_S \Delta}$; additionally, the initial phase increases as $\Delta$ is increased.

The transfer function is easily obtained from Equation 18 using Equation 27 for $B_1$ and the fact that $C_1 = 0$ for white noise:

$$H(j\omega) = \frac{e^{(-\alpha_S + j\omega_S)\Delta} (1 - e^{\alpha_S - \mu})}{1 - e^{-\mu} e^{j(\omega_S - \omega)}}.$$
(29)

## SEPARATION OF SIGNALS FROM NOISE USING DELAY PARAMETER, $\Delta$

In discrete Wiener prediction theory, we estimate the value of a discrete time series $\Delta$ samples into the future based on the knowledge of its present and past values. That is, based on a knowledge of the sequence* $\{x(k), x(k - 1), \ldots, x(k - L + 1)\}$, we form an MMSE prediction of $x(k + \Delta)$. This is equivalent also to predicting the present value of the time series $x(k)$ based on the delayed sequence $\{x(k - \Delta), x(k - \Delta - 1), \ldots, x(k - \Delta - L + 1)\}$. In this section we show that we are able to separate the BL signal from the lowpass noise background using a variable prediction distance $\Delta$. A requirement is some a priori knowledge of the expected signal and noise correlation distances (or, equivalently, signal and noise 3 dB bandwidths).

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*For filters not constrained to be finite length, this sequence extends an infinite distance in the past: $\{x(k), x(k - 1), \ldots\}$.

Using Equations 14 and 16 we see that the expression for the impulse response for long filter lengths may be rewritten to reflect the dependence of the $B_1$ and $C_1$ coefficients upon $\Delta$:

$$h_\Delta(k) = B_1(\Delta) \, e^{-\mu k} \, e^{j\theta_1 k} + C_1(\Delta) \, \delta(k) \tag{30}$$

$$k = 0, \ldots, L - 1; L \to \infty,$$

where $B_1(\Delta)$ and $C_1(\Delta)$ are given by Equations 14a and 14b, respectively. From Equations 14 we see that for fixed signal and noise parameters ($\alpha_N$, $\alpha_S$, $\theta_1$, $\omega_S$, $\mu$) the coefficients $B_1(\Delta)$ and $C_1(\Delta)$ (and hence the filter impulse response) are functions of the prediction distance $\Delta$. Since $B_1(\Delta)$ is associated with the BL signal characteristics of the optimal filter and $C_1(\Delta)$ is associated with the noise contribution, we can enhance the BL signal properties by choosing a value of $\Delta$ for which $B_1(\Delta)$ dominates $C_1(\Delta)$.

This can be seen clearly in the following example. Consider the input power spectrum as shown in Figure 1 and suppose we desire to use the MMSE filter impulse response to estimate the center frequency of the BL signal. For the parameters given in Figure 1, the MMSE filter has the impulse response given by

$$h_\Delta(k) = B_1(\Delta) \, [.772 \, e^{j(.490)\pi}]^k + C_1(\Delta) \, \delta(k) \tag{31}$$

where from Equation 7, the $z_m = z_1, z_2$ are found to be

$$z_1 = .772 \, e^{j(.490)\pi} \tag{32a}$$

$$z_2 = 1.295 \, e^{j(.490)\pi}. \tag{32b}$$

From Equation 31 we can see that the impulse response now becomes a function of $\Delta$ via the $B_1(\Delta)$ and $C_1(\Delta)$ dependence. Figure 2 shows the relative magnitudes of $B_1(\Delta)$ and $C_1(\Delta)$ as a function of $\Delta$ for the signal and noise parameters given in Figure 1. From Figure 2 we see that for small values of delay ($\Delta < 7$), the noise coefficient $C_1(\Delta)$ is much larger in magnitude than the signal coefficient $B_1(\Delta)$. The resulting impulse response would then be strongly influenced by the noise component, producing a strong impulse for the $k = 0$ weight. This is shown explicitly in Figure 3a, which is the real part of the complex weight vector for the value $\Delta = 1$. There is a strong impulsive value at $k = 0$, after which the signal components take the form of a damped complex exponential. Note, however, that for $\Delta \geqslant 7$ in Figure 2 the $B_1(\Delta)$ coefficient is the larger, which signifies the signal components should dominate the impulse response. This is clearly seen in Figure 3b, which illustrates the almost complete disappearance of the impulse-type value at $k = 0$ for $\Delta = 7$. The weight vector is very nearly a noise-free damped complex exponential which should lead to good frequency estimation properties.

Figure 4 then presents the magnitude square of the transfer functions, $|H(\omega)|^2$, of the weight vectors shown in Figure 3. For $\Delta = 1$ (Figure 4a) the transfer function is very much dominated by the noise component and a highly inaccurate estimation of the BL signal results. As $\Delta$ is increased to $\Delta = 7$, it is possible to estimate frequency better from the transfer function (Figure 4b). One effect, however, (seen from Figure 4b) is that the resultant BL signal appearing in the transfer function is broader than the original BL signal shown in Figure 1. However, the spectral peak of the transfer function is at the correct center frequency $\omega_S = .50\pi$ of the BL signal.
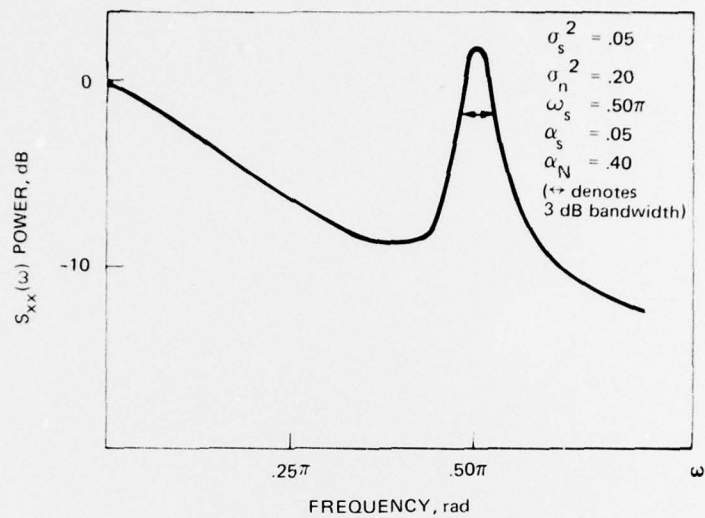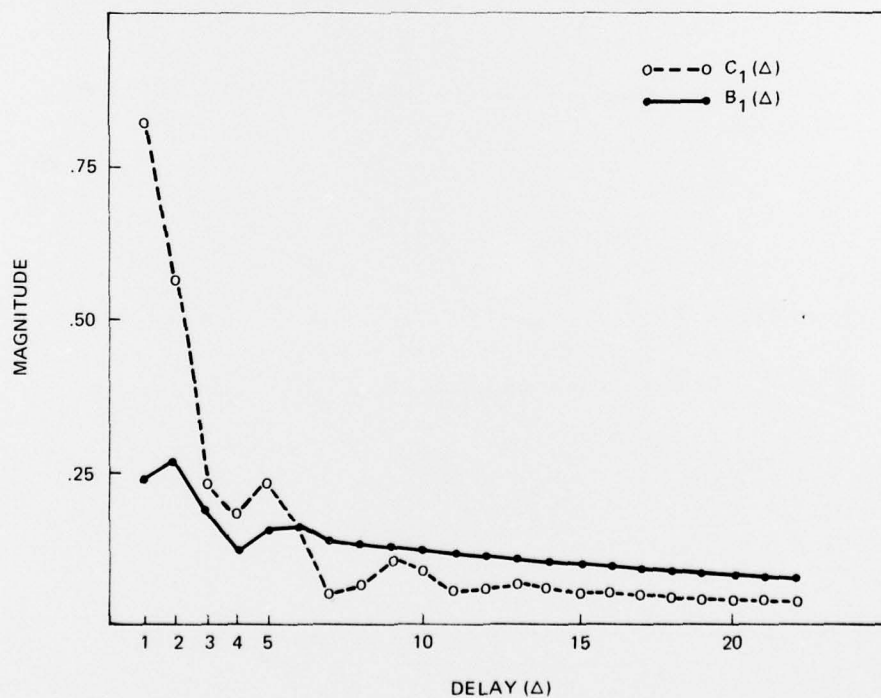
Figure 1. Input power spectrum for BL complex signal in lowpass noise.



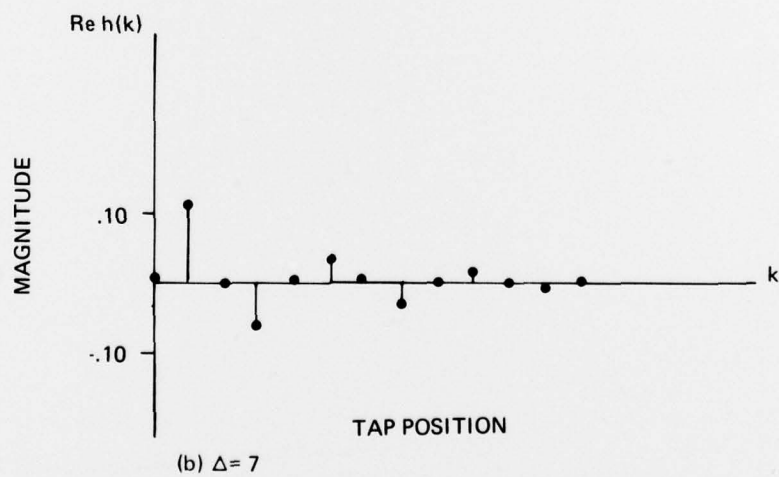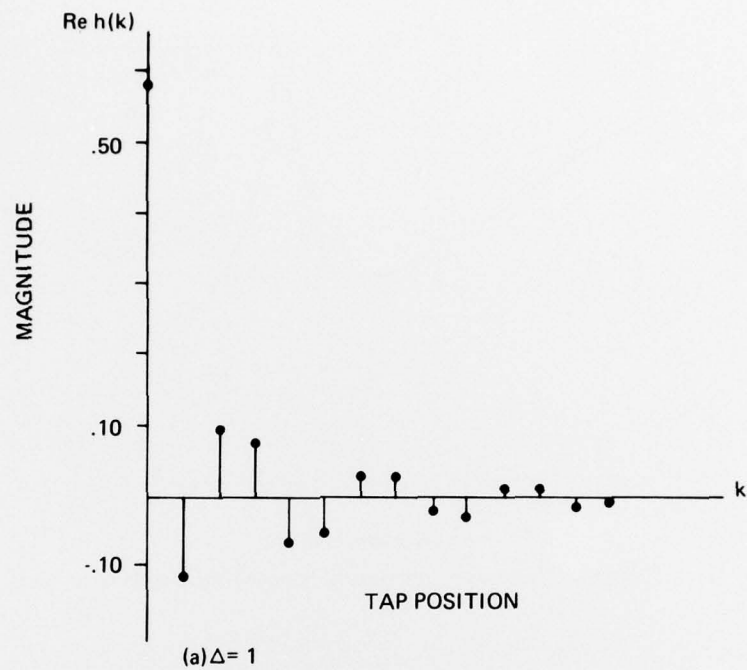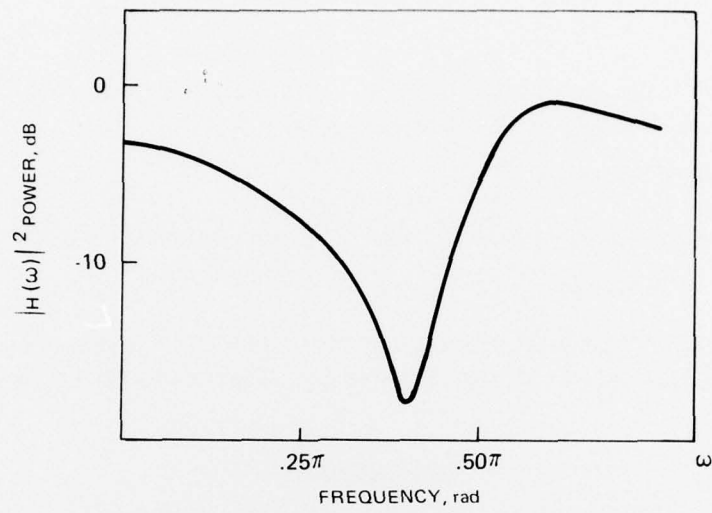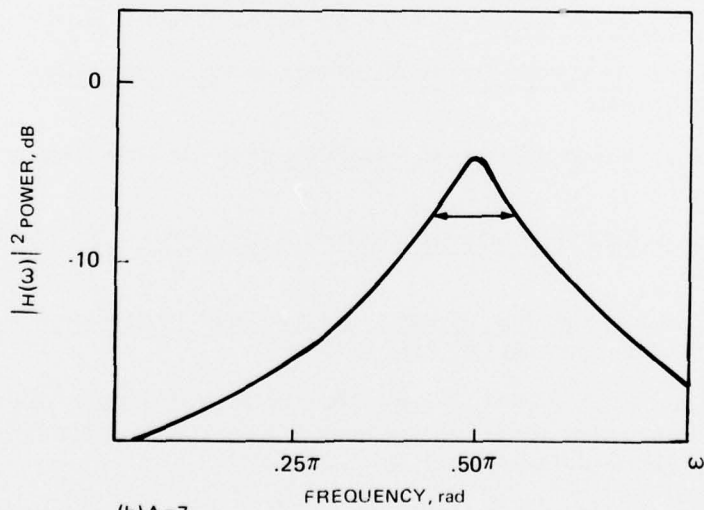Figure 2. Magnitude of $B_1$ and $C_1$ coefficients.

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Figure 3. Optimal filter impulse response for input spectrum of Figure 1.

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Figure 4. Transfer function of optimal filter.

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By increasing $\Delta$ to a value greater than the correlation distance of the lowpass noise, the effects of the noise upon the MMSE filter response may be diminished considerably. However, as long as $\Delta$ is not increased past the correlation distance of the BL signal, the filter response still retains good characteristics for detecting and estimating the center frequency of the BL signal. If we define $\omega_0$ as the 3 dB bandwidth of a BL process, $\omega_0 = 2\alpha$ leads to a process autocorrelation function of the form $\phi(\ell) = \exp[-\alpha|\ell|]$. This leads to defining the correlation distance, $\Gamma$, of the process as

$$\Gamma = 1/\alpha, \tag{33}$$

so that when $\ell = \Gamma$, the value of $\phi(\ell) = e^{-1}$. For the lowpass noise process of Figure 1, the correlation distance of the noise, $\Gamma_N$, then is given by

$$\Gamma_N = 1/\alpha_N = 2.5 \text{ samples,}$$

whereas the correlation distance of the BL signal, $\Gamma_S$, is much longer and given by

$$\Gamma_S = 1/\alpha_S = 20 \text{ samples.}$$

Thus, the value of $\Delta = 7$ samples chosen from Figure 2 falls safely within the range $\Gamma_N < \Delta < \Gamma_S$ required to give good signal characteristics in the transfer function.

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# APPENDIX
## AUTOCORRELATION FOR CORRELATED NOISE AND BL COMPLEX LINE

The autocorrelation function $\phi(\ell)$ and power spectral density $S(\omega)$ are related by the inversion integral

$$\phi(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega)\, e^{j\omega\ell}\, d\omega \ . \tag{A1}$$

Now consider the discrete form $S(z)$ related to $S(\omega)$ through the transformation (for $T=1$) $z = e^{j\omega}$. The new parameters in Equation A1 may be found by the following relations:

$$dz = j\, e^{j\omega}\, d\omega \rightarrow d\omega = -j\, z^{-1}\, dz$$

$$\left.\begin{array}{l} \omega = -\pi \\ \omega = \pi \end{array}\right\} \rightarrow \left\{\begin{array}{l} z = e^{-j\pi} \\ z = e^{j\pi} \ . \end{array}\right.$$

The last relation implies the path of integration in the z-plane is counterclockwise around the unit-circle. Thus, in the z-plane Equation A1 becomes

$$\phi(\ell) = \frac{1}{2\pi j} \oint_c S(z)\, z^{\ell-1}\, dz$$

$$= \sum \operatorname{Res} S(z)\, z^{\ell-1} \Big|_{\text{poles of } S(z)} \ . \tag{A2}$$

Assume first that $S(z)$ is the lowpass noise spectrum given by $S_{NN}(z)$:

$$S_{NN}(z) = \frac{2\,\sigma_N^2\, e^{-\alpha_N} \sinh \alpha_N}{\left(z - e^{-\alpha_N}\right)\left(z^{-1} - e^{-\alpha_N}\right)} \ . \tag{A3}$$

Applying the inversion integral (Equation A2), the autocorrelation function $\phi_{NN}(\ell)$ becomes:

$$\phi_{NN}(\ell) = \frac{\left(2\,\sigma_N^2 \sinh \alpha_N\right) z^\ell}{\left(e^{\alpha_N} - z\right)} \Big|_{z\, =\, e^{-\alpha_N}} \ , \quad \ell \geqslant 0 \ . \tag{A4}$$

Evaluating Equation A4:

$$\phi_{NN}(\ell) = \sigma_N^2\, e^{-\alpha_N \ell} \ , \quad \ell \geqslant 0 \ . \tag{A5}$$

A similar evaluation for $\ell < 0$ gives

$$\phi_{NN}(\ell) = \sigma_N^2\, e^{\alpha_N \ell} \ , \quad \ell < 0 \ . \tag{A6}$$

The combination of Equations A5 and A6 give

$$\phi_{NN}(\ell) = \sigma_N^2\, e^{-\alpha_N |\ell|} \tag{A7}$$

which is the desired autocorrelation function.

Next consider the complex BL signal with the spectrum $S_{ss}(z)$ given by:

$$S_{ss}(z) = \frac{2 \sigma_s^2 e^{-\alpha_s} \sinh \alpha_s}{\left(z - e^{-\alpha_s + j\omega_s}\right) \left(z^{-1} - e^{-\alpha_s - j\omega_s}\right)} \ . \tag{A8}$$

Again applying the inversion integral gives the autocorrelation function $\phi_{ss}(\ell)$:

$$\phi_{ss}(\ell) = \left. \frac{\left(2 \sigma_s^2 \sinh \alpha_s\right) z^\ell e^{j\omega_s}}{\left(e^{\alpha_s + j\omega_s} - z\right)} \right|_{z = e^{-\alpha_s + j\omega_s}} , \quad \ell \geqslant 0 \ . \tag{A9}$$

Upon evaluating, Equation A9 reduces to

$$\phi_{ss}(\ell) = \sigma_s^2 e^{-\alpha_s \ell} e^{j\omega_s \ell} , \quad \ell \geqslant 0 \ . \tag{A10}$$

A similar solution for $\ell < 0$ gives

$$\phi_{ss}(\ell) = \sigma_s^2 e^{\alpha_s \ell} e^{j\omega_s \ell} , \quad \ell < 0 \tag{A11}$$

and combining Equations A10 and A11 gives the required autocorrelation function for the BL signal:

$$\phi_{ss}(\ell) = \sigma_s^2 e^{-\alpha_s |\ell|} e^{j\omega_s \ell} \ . \tag{A12}$$

The linearity of autocorrelation functions now allows us to find the autocorrelation function, $\phi_{xx}(\ell)$, of an input process consisting of a complex BL signal in lowpass noise by simple addition of Equations A7 and A12:

$$\phi_{xx}(\ell) = \phi_{ss}(\ell) + \phi_{NN}(\ell) \ .$$

The linearity of the z-transform operator then gives

$$S_{xx}(z) = S_{NN}(z) + S_{ss}(z) \ . \tag{A13}$$

Performing the addition in Equation A13 gives

$$S_{xx}(z) = \frac{2 \sigma_N^2 e^{-\alpha_N} \sinh \alpha_N}{\left(z - e^{-\alpha_N}\right) \left(z^{-1} - e^{-\alpha_N}\right)} + \frac{2 \sigma_s^2 e^{-\alpha_s} \sinh \alpha_s}{\left(z - e^{-\alpha_s + j\omega_s}\right) \left(z^{-1} - e^{-\alpha_s - j\omega_s}\right)} \ . \tag{A14}$$

After considerable algebra, the final form of the input spectrum becomes

$$S_{xx}(z) = \frac{(-2z) N(z)}{D(z)} \tag{A15}$$

where

$$N(z) = \sigma_N^2 \sinh \alpha_N \left( z^2 - 2z \, e^{j\omega_s} \cosh \alpha_s + e^{j2\omega_s} \right)$$

$$+ \sigma_s^2 \, e^{j\omega_s} \sinh \alpha_s \left( z^2 - 2z \cosh \alpha_N + 1 \right) \tag{A16}$$

$$D(z) = \left( z - e^{-\alpha_N} \right) \left( z - e^{\alpha_N} \right) \left( z - e^{-\alpha_s + j\omega_s} \right) \left( z - e^{\alpha_s + j\omega_s} \right) . \tag{A17}$$

The poles of $S_{XX}(z)$ are the roots of the Equation $D(z) = 0$ and are the pole-pairs of each separate spectrum from Equations A3 and A8. The roots of the Equation $N(z)$ are the non-trivial zeros of $S_{XX}(z)$. Additionally, there is a zero at the origin which does not affect computation of the spectrum or the solution for the weight vector.